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A LIMIT THEOREM FOR THE MOMENTS OF SUMS OF INDEPENDENT RANDOM VARIABLES

BY

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ABSTRACT

For $n \geqq 1$, let $S_n = \Sigma X_{n,i}$ $(1 \leqq i \leqq r_n < \infty)$, where the summands of S_n are independent random variables having medians bounded in absolute value by a finite number which is independent of n. Let f be a nonnegative function on $(-\infty, \infty)$ which vanishes and is continuous at the origin, and which satisfies, for some $\alpha > 0$, $f(x) \leqq f(tx) \leqq t^\alpha f(x)$ for all $t \geqq 1$ and all values of x.

THEOREM. *For centering constants c_n, let $S_n - c_n$ converge in distribution to a random variable S.* (A) *In order that $Ef(S_n - c_n)$ converge to a limit L, it is necessary and sufficient that there exist a common limit*

$$R = \lim_{t \to \infty} \overline{\lim_{n \to \infty}} \sum_{i=1}^{r_n} \int f(X_{n,i}) I(|X_{n,i}| > t).$$

(B) *If L exists, then $L < \infty$ if and only if $R < \infty$, and when L is finite, $L = Ef(S) + R$.*

Applications are given to infinite series of independent random variables, and to normed sums of independent, identically distributed random variables.

§0. Introduction

A weak moment function is a nonnegative function f on $(-\infty, \infty)$ which vanishes and is continuous at the origin, which is monotone on each of the intervals $(-\infty, 0]$ and $[0, \infty)$, and which satisfies for some $\alpha > 0$

$$(0.1) \qquad f(tx) \leqq t^\alpha f(x) \qquad t \geqq 1, \quad -\infty < x < \infty.$$

An array of random variables is median-bounded if some finite interval contains a median of each variable of the array.

We prove the following limit theorem, in which $S_n = \Sigma X_{n,i}$ $(1 \leqq i \leqq r_n)$,

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$\{c_n, n \geqq 1\}$ is a sequence of constants, S is a random variable, and $\xrightarrow{\mathscr{D}}$ denotes convergence in distribution.

THEOREM 0. *Let the array $\{X_{n,i}; 1 \leqq i \leqq r_n\}$ be row-wise independent and median-bounded, and let $S_n - c_n \xrightarrow{\mathscr{D}} S$. Let f be a weak moment function.*

(A) *In order that $Ef(S_n - c_n)$ converge to a limit L, it is necessary and sufficient that there exists a common limit*

$$R = \lim_{t\to\infty} \overline{\lim_{n\to\infty}} \sum_{i=1}^{r_n} \int_{|X_{n,i}|>t} f(X_{n,i}). \tag{0.2}$$

(B) *Suppose that the limit L exists. Then $L < \infty$ if and only if $R < \infty$, and when L is finite,*

$$L = Ef(S) + R.$$

By the existence of a "common limit" in (0.2), we mean, of course, that $\lim_{t\to\infty}\underline{\lim}_{n\to\infty}$ and $\lim_{t\to\infty}\overline{\lim}_{n\to\infty}$ of the sum in (0.2) are equal, with R denoting their common value.

In §1 we prove some preliminary lemmas, in §2 we prove Theorem 0, and in §3 we consider applications to infinite series of independent random variables and to normed sums of independent, identically distributed random variables.

§1. Preliminary lemmas

LEMMA 1.1. *Let f be a weak moment function, let $\alpha > 0$ satisfy* (0.1), *and let x and y be any real numbers. Then*

(i) $f^{1/\alpha}(x+y) \leqq f^{1/\alpha}(x) + f^{1/\alpha}(y)$.

(ii) *Let $0 < \varepsilon < 1$ and $|x| \leqq \varepsilon |y|$. Then*

$$(1-\varepsilon)^\alpha f(y) \leqq f(x+y) \leqq (1+\varepsilon)^\alpha f(y).$$

PROOF. (i) If $x + y = 0$, the inequality is obvious. The cases $x + y < 0$ and $x + y > 0$ are similar, and we consider the case $x + y > 0$ only. Suppose first that $x > 0$ and $y > 0$.

The inequality (0.1) may be written

$$s^\alpha f(z) \leqq f(sz), \qquad 0 < s \leqq 1, \quad -\infty < z < \infty. \tag{1.1}$$

Letting $s = x/x + y$, $z = x + y$, and taking αth roots yields

$$x(x+y)^{-1} f^{1/\alpha}(x+y) \leqq f^{1/\alpha}(x).$$

Interchanging x and y and summing the results yields (i).

Next suppose that one of the variables, say x, is nonpositive. Then $0 < x + y \leqq y$, and the inequality follows from the monotonicity of $f^{1/\alpha}$ on $[0, \infty)$.

(ii) Suppose first that $y \geqq 0$. Then $0 \leqq (1-\varepsilon)y \leqq x + y \leqq (1+\varepsilon)y$, implying $f\{(1-\varepsilon)y\} \leqq f(x+y) \leqq f\{(1+\varepsilon)y\}$, and applications of (1.1) and (0.1) yield the asserted inequalities. The case $y \leqq 0$ is similar.

LEMMA 1.2. *For any integer $m \geqq 1$ there exists a polynomial $\psi_m(x)$ having nonnegative coefficients such that $\psi_m(0) = 0$ and such that for any finite sequence $\{X_i\}$ of independent random variables having finite mth moments*

$$\left| E\Big(\sum_i X_i\Big)^m \right| \leqq \sum_i |EX_i^m| + \psi_m\Big(\max_{1 \leqq k < m} \sum_i |EX_i^k|\Big). \tag{1.2}$$

(The argument of ψ_m in (1.2) is intended to be 0 when $m = 1$.)

PROOF. We shall show that the sequence $\{\psi_m, m \geqq 1\}$ defined recursively by

$$\psi_1(x) = 0, \qquad \psi_m(x) = \sum_{a=1}^{m-1} \binom{m}{a} x\{x + \psi_a(x)\}$$

does the job. We proceed by induction on m.

$\psi_1(x)$ certainly has the required property. Suppose that $\psi_a(x)$ has it when $1 \leqq a < m$. To show that ψ_m has it, it suffices to show that (1.2) holds for sequences of the form $\{X_i, 0 \leqq i \leqq r\}$ where $X_0 = 0$ and $r \geqq 1$. Let

$$T_i = \sum_{j=0}^{i} X_j, \qquad 0 \leqq i \leqq r;$$

$$u = \sum_{i=0}^{r} |EX_i^m|; \qquad v = \max_{1 \leqq k < m} \sum_{i=0}^{r} |EX_i^k|.$$

We must show that

$$|ET_r^m| \leqq u + \psi_m(v). \tag{1.3}$$

Now

$$(x+y)^m - y^m = x^m + \sum_{a=1}^{m-1} \binom{m}{a} x^{m-a} y^a.$$

Letting $x = X_{i+1}$, $y = T_i$, and taking expectations yields

$$ET_{i+1}^m - ET_i^m = EX_{i+1}^m + \sum_{a=1}^{m-1} \binom{m}{a} EX_{i+1}^{m-a} ET_i^a.$$

By the induction hypotheses, for $1 \leqq a \leqq m-1$ we have

$$|ET_i^a| \leqq \sum_{j=0}^{j} |EX_j^a| + \psi_a\left(\max_{1\leqq k<a} \sum_{j=0}^{i} |EX_j^k|\right)$$

$$\leqq v + \psi_a(v).$$

Hence

$$ET_{i+1}^m - ET_i^m \leqq |EX_{i+1}^m| + \sum_{a=1}^{m-1} \binom{m}{a} |EX_{i+1}^{m-a}| \{v + \psi_a(v)\}.$$

Summing over $0 \leqq i \leqq r-1$ leads (crudely) to

$$ET_r^m \leqq u + \sum_{a=1}^{m-1} \binom{m}{a} v\{v + \psi_a(v)\} = u + \psi_m(v).$$

The same argument applied to the sequence $\{-X_i, 0 \leqq i \leqq r\}$ leads to (1.3) and completes the proof.

From now on, Σ denotes $\Sigma_{i=1}^{r_n}$ and $\bigcup$ denotes $\bigcup_{i=1}^{r_n}$.

TRUNCATION NOTATION. We denote

$$X_{n,i}(t) = X_{n,i} I_{(|X_{n,i}| \leqq t)}, \qquad X'_{n,i}(t) = X_{n,i} I_{(|X_{n,i}| > t)};$$

$$S_n(t) = \sum X_{n,i}(t) \qquad S'_n(t) = \sum X'_{n,i}(t);$$

so

$$S_n = S_n(t) + S'_n(t).$$

LEMMA 1.3. *Let the array* $\{X_{n,i}; 1 \leqq i \leqq r_n, n \geqq 1\}$ *be row-wise independent and median-bounded, and let* $S_n - c_n \xrightarrow{\mathcal{D}} S$. *Then*

(i) $\lim_{t\to\infty} \sup_{n\geqq 1} \Sigma P\{|X_{n,i}| > t\} = 0$ *and for all values of t sufficiently large,*

(ii) $\sup_{n\geqq 1} \mathrm{Var} S_n(t) < \infty$,

(iii) $\sup_{n\geqq 1} |ES_n(t) - c_n| < \infty$.

PROOF. (i) Let $d_{n,i}$ be the median of $X_{n,i}$ of smallest absolute value. Then for some $C < \infty$, $|d_{n,i}| < C$ for every i and n. By symmetrization inequalities† in [1], we have for every $t > C$

$$\sum P\{|X_{n,i}| > t\} \leqq \sum P\{|X_{n,i} - d_{n,i}| > t - C\} \leqq 2 \sum P\{|X_{n,i}^s| > t - C\}$$

$$\leqq -2\log[1 - 2P\{|S_n^s| > t - C\}], \tag{1.4}$$

† See page 149 of [1], inequality (5.8) and the obvious generalization of (5.10) to nonidentically distributed symmetric variables.

where s denotes symmetrization. Now $S_n^s = (S_n - c_n)^s \xrightarrow{\mathcal{D}} S^s$. Hence the last term of (1.4) converges to 0 as $t \to \infty$ uniformly in n, and (i) follows.

By (i), for all values of t sufficiently large,

$$\sum P\{|X_{n,i}| > t\} \leqq \tfrac{1}{4} \qquad \text{for all } n \geqq 1. \tag{1.5}$$

We shall show that (ii) and (iii) hold for such values of t. We fix such a value.

(ii) Let $\sigma_n^2 = \operatorname{Var} S_n(t)$. We first show that it is impossible both that $\sigma_n \to \infty$ and that $c_n \leqq ES_n(t)$ for every $n \geqq 1$. Indeed, suppose the contrary. Then $\sigma_n^{-1}\{S_n(t) - ES_n(t)\} \xrightarrow{\mathcal{D}} Z$, where Z has a standard normal distribution, and also, for any $M > 0$,

$$\begin{aligned} P\{S_n - c_n \leqq M\} &\leqq P\{S_n - ES_n(t) \leqq M\} \leqq P\{S_n \neq S_n(t)\} + P\{S_n(t) - ES_n(t) \leqq M\} \\ &\leqq \tfrac{1}{4} + P\{\sigma_n^{-1}[S_n(t) - ES_n(t)] \leqq \sigma_n^{-1}M\} \end{aligned}$$

where we applied (1.5) in the last step. Letting $n \to \infty$ yields $P\{S \leqq M\} \leqq \frac{1}{4} + P\{Z \leqq 0\} = 3/4$ for arbitrarily large values of M, and we arrive at a contradiction.

Applying what we have just proved to all subarrays of $\{X_{n,i}\}$ and of $\{-X_{n,i}\}$ yields (ii).

(iii) We first show the impossibility of $ES_n(t) - c_n \to \infty$. Suppose that this convergence holds. By (ii), some $0 < B < \infty$ satisfies $\sup_{n \geqq 1} \operatorname{Var} S_n(t) \leqq \frac{1}{4}B^2$. For any $M > 0$ and for all values of n sufficiently large, we have

$$\begin{aligned} P\{S_n - c_n \leqq -B + M\} &\leqq P\{S_n - c_n \leqq -B + ES_n(t) - c_n\} = P\{S_n - ES_n(t) \\ &\qquad \leqq -B\} \\ &\leqq P\{S_n \neq S_n(t)\} + P\{S_n(t) - ES_n(t) \leqq -B\} \\ &\leqq \tfrac{1}{4} + B^{-2}\operatorname{Var} S_n(t) \leqq \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}. \end{aligned}$$

Letting $n \to \infty$, we obtain $P\{S \leqq -B + M\} \leqq \frac{1}{2}$ for arbitrarily large values of M, which is a contradiction.

Applying the foregoing result to all subarrays of $\{X_{n,i}\}$ and of $\{-X_{n,i}\}$ yields (iii).

§2. The proof of Theorem 0

We now prove Theorem 0. f is obviously continuous everywhere, and it follows that if $P\{|S| = t\} = 0$, then

$$\int_{|S_n - c_n| \leqq t} f(S_n - c_n) \to \int_{|S| \leqq t} f(S) \qquad \text{as } n \to \infty.$$

From here it is clear that Theorem 0 is true if we replace (0.2) by

$$Q = \lim_{t\to\infty} \overline{\underline{\lim}}_{n\to\infty} \int_{|S_n - c_n|>t} f(S_n - c_n)$$

and if all further occurrences of R are replaced by Q.

Hence, it remains to prove that

(2.1) $\qquad Q$ exists if and only if R exists, and then $Q = R$.

Let

$$Q_n(t) = \int_{|S_n - c_n|>t} f(S_n - c_n), \qquad R_n(t) = \sum \int_{|X_{n,i}|>t} f(X_{n,i}),$$

$$\underline{Q} = \lim_{t\to\infty} \underline{\lim}_{n\to\infty} Q_n(t), \qquad \bar{Q} = \lim_{t\to\infty} \overline{\lim}_{n\to\infty} Q_n(t),$$

and similarly define $\underline{R}$ and $\bar{R}$ with respect to $R_n(t)$.

To prove (2.1), it suffices to prove that

$$\underline{Q} = \underline{R} \quad \text{and} \quad \bar{Q} = \bar{R}.$$

We shall first prove that $\underline{Q} \leqq \underline{R}$ and $\bar{Q} \leqq \bar{R}$, and later that $\underline{Q} \geqq \underline{R}$ and $\bar{Q} \geqq \bar{R}$.

$\underline{Q} \leqq \underline{R}$ and $\bar{Q} \leqq \bar{R}$. We shall give explicitly the proof of $\underline{Q} \leqq \underline{R}$ only. This proof, however, may be converted into a proof of $\bar{Q} \leqq \bar{R}$ by systematically replacing each occurrence of $\underline{Q}$, $\underline{R}$, and $\underline{\lim}$ in it by $\bar{Q}$, $\bar{R}$, and $\overline{\lim}$ respectively.

We may assume that $\underline{R} < \infty$, since otherwise the inequality is trivial. We may also assume that $f(-1) \leqq 1$ and $f(1) \leqq 1$, since multiplying f by a positive constant affects neither the weak moment property nor the inequality to be proved. Letting $x = \pm 1$ in (0.1), we then have

$$f(\tau) \leqq |\tau|^{\alpha}, \qquad -\infty < \tau < \infty.$$

For any positive numbers s, t and ε, we have

(2.2) $$Q_n(s) \leqq \int_{A_n(\varepsilon,s,t)} f(S_n - c_n) + \int_{B_n(\varepsilon,t)} f(S_n - c_n)$$

where

$$A_n(\varepsilon, s, t) = \{|S_n - c_n| > s,\ |S_n(t) - c_n| > \varepsilon |S_n'(t)|\},$$

$$B_n(\varepsilon, t) = \{|S_n(t) - c_n| \leqq \varepsilon |S_n'(t)|\}.$$

Let p be an even integer greater than α. We first show that

$$\int_{A_n(\varepsilon,s,t)} f(S_n - c_n) \leqq s^{\alpha-p}(1+\varepsilon^{-1})^p E\{S_n(t) - c_n\}^p, \tag{2.3}$$
$$s \geqq 1, \quad t > 0, \quad \varepsilon > 0.$$

Let $s \geqq 1$. On the set $A_n(\varepsilon, s, t)$ we have $|S_n - c_n| > s \geqq 1$, so on this set

$$f(S_n - c_n) \leqq |S_n - c_n|^\alpha \leqq s^{\alpha-p}(S_n - c_n)^p.$$

Moreover, on this set $|S_n'(t)| \leqq \varepsilon^{-1}|S_n(t) - c_n|$, and hence $|S_n - c_n| \leqq (1+\varepsilon^{-1})|S_n(t) - c_n|$. It follows that

$$f(S_n - c_n) \leqq s^{\alpha-p}(1+\varepsilon^{-1})^p\{S_n(t) - c_n)\}^p$$

on $A_n(\varepsilon, s, t)$, and (2.3) follows.

Let (ii) and (iii) of Lemma 1.3 hold for all $t > t_0$. We now show that

$$\sup_{n \geqq 1} E\{S_n(t) - c_n\}^p < \infty, \qquad t > t_0. \tag{2.4}$$

By (iii) of Lemma 1.3, to prove (2.4) it suffices to show that

$$\sup_{n \geqq 1} E\{S_n(t) - ES_n(t)\}^p < \infty, \qquad t > t_0. \tag{2.5}$$

We fix a value of $t > t_0$ and let $Y_{n,i} = X_{n,i}(t) - EX_{n,i}(t)$. Then $S_n(t) - ES_n(t) = \Sigma\, Y_{n,i}$, so by Lemma 1.2, to prove (2.5), it suffices to prove that

$$\sup_{n \geqq 1} \sum |EY_{n,i}^k| < \infty \qquad \text{for each } k \geqq 1.$$

When $k = 1$, the inequality is clear. When $k = 2$ it just asserts that $\sup_{n \geqq 1} \operatorname{Var} S_n(t) < \infty$, which holds by hypothesis. If $k > 2$, then as $|Y_{n,i}| < 2t$, we have

$$|EY_{n,i}^k| \leqq E|Y_{n,i}^k| \leqq (2t)^{k-2} EY_{n,i}^2,$$

and the case $k > 2$ follows from the case $k = 2$.

From (2.4) we see that for any fixed values of $t > t_0$ and $\varepsilon > 0$, the right side of (2.3) is small uniformly in n when s is large, and from (2.2) we easily obtain

$$Q \leqq \varliminf_{n\to\infty} \int_{B_n(\varepsilon,t)} f(S_n - c_n) \qquad t > t_0, \quad \varepsilon > 0. \tag{2.6}$$

As $|S_n(t) - c_n| \leqq \varepsilon |S_n'(t)|$ on $B_n(\varepsilon, t)$, we see from Lemma 1.1 (ii) that for $0 < \varepsilon < 1$,

$$\int_{B_n(\varepsilon,t)} f(S_n - c_n) \leqq (1+\varepsilon)^\alpha Ef\{S_n'(t)\}.$$

Taking $\underline{\lim}_{n\to\infty}$ of each side, applying (2.6), and then letting $\varepsilon \to 0$ and $t \to \infty$ yields

$$Q \leqq \overline{\lim_{t\to\infty}}\ \underline{\lim_{n\to\infty}}\ Ef\{S'_n(t)\},$$

and it remains to show that

$$\overline{\lim_{t\to\infty}}\ \underline{\lim_{n\to\infty}}\ Ef\{S'_n(t)\} \leqq \underset{\sim}{R}. \tag{2.7}$$

Let m be an integer satisfying $m \geqq \alpha$. From Lemma 1.1 (i) and induction, we obtain for $t > 0$

$$f^{1/m}\{S'_n(t)\} \leqq \sum f^{1/m}\{X'_{n,i}(t)\}$$

or

$$f\{S'_n(t)\} \leqq \left[\sum f^{1/m}\{X'_{n,i}(t)\}\right]^m.$$

Hence by Lemma 1.2 (or trivially if $Ef\{X'_{n,i}(t)\} = \infty$ for some i)

$$Ef\{S'_n(t)\} \leqq \sum Ef\{X'_{n,i}(t)\} + \psi_m\left[\max_{1\leqq k<m} \sum Ef^{k/m}\{X'_{n,i}(t)\}\right]. \tag{2.8}$$

Now let $M > 1$. For any number $x \geqq 0$ and for $1 \leqq k < m$

$$\begin{aligned} x^k &\leqq M^k I(0 < x < M) + M^{k-m}x^m I(x \geqq M) \\ &\leqq M^{m-1}I(0 < x) + M^{-1}x^m. \end{aligned}$$

Letting $x = f^{1/m}\{X'_{n,i}(t)\}$, it follows that

$$f^{k/m}\{X'_{n,i}(t)\} \leqq M^{m-1}I(|X_{n,i}| > t) + M^{-1}f\{X'_{n,i}(t)\}.$$

Taking expectations and summing yields

$$\sum Ef^{k/m}\{X'_{n,i}(t)\} \leqq M^{m-1}\gamma(t) + M^{-1}\sum Ef\{X'_{n,i}(t)\}, \tag{2.9}$$

where $\gamma(t) = \sup_{n\geqq 1}\Sigma P\{|X_{n,i}| > t\}$. Now $\Sigma Ef\{X'_{n,i}(t)\} = R_n(t)$, so by (2.8) and (2.9)

$$Ef\{S'_n(t)\} \leqq R_n(t) + \psi_m\{M^{m-1}\gamma(t) + M^{-1}R_n(t)\}.$$

From Lemma 1.3, $\gamma(t) \to 0$ as $t \to \infty$; it follows that

$$\overline{\lim_{t\to\infty}}\ \underline{\lim_{n\to\infty}}\ Ef\{S'_n(t)\} \leqq \underset{\sim}{R} + \psi_m\{M^{-1}\underset{\sim}{R}\}$$

for every $M > 1$, and (2.7) follows.

We now prove the remaining pair of inequalities $Q \geqq \underset{\sim}{R}$ and $\bar{Q} \geqq \bar{R}$. Fix $0 < \varepsilon < 1$, and let

$$M_n = \max\{|X_{n,i}|: 1 \leqq i \leqq r_n\},$$

$$M_{n,i} = \max\{|X_{n,j}|: 1 \leqq j \leqq r_n, j \neq i\},$$

$$A_{n,i}(t) = \{|S_n - c_n - X_{n,i}| < \varepsilon t,\ M_{n,i} < t\},$$

$$B_{n,i}(t) = \{|X_{n,i}| > t\} \cap A_{n,i}(t)\},$$

$$C_n(t) = \{|S_n - c_n| < t,\ M_n < t\}.$$

We first prove the following chain of inequalities:

$$Q_n\{(1-\varepsilon)t\} = \int_{|S_n - c_n| > (1-\varepsilon)t} f(S_n - c_n) \geqq \int_{\cup B_{n,i}(t)} f(S_n - c_n) \tag{2.10}$$

$$= \sum \int_{B_{n,i}(t)} f(S_n - c_n) \geqq (1-\varepsilon)^\alpha \sum \int_{B_{n,i}(t)} f(X_{n,i}) \tag{2.11}$$

$$= (1-\varepsilon)^\alpha \sum \left[P\{A_{n,i}(t)\} \int_{|X_{n,i}|>t} f(X_{n,i}) \right] \tag{2.12}$$

$$\geqq (1-\varepsilon)^\alpha P\{C_n(\varepsilon t/2)\} R_n(t). \tag{2.13}$$

On the set $B_{n,i}(t)$ we have $|S_n - c_n - X_{n,i}| < \varepsilon |X_{n,i}|$ and two applications of Lemma 1.1 (ii) yield

$$f(S_n - c_n) \geqq (1-\varepsilon)^\alpha f(X_{n,i}) \quad \text{on } B_{n,i}(t), \tag{2.14}$$

$$|S_n - c_n| \geqq (1-\varepsilon)|X_{n,i}| \quad \text{on } B_{n,i}(t). \tag{2.15}$$

Now $|X_{n,i}| > t$ on $B_{n,i}(t)$, so by (2.15) $|S_n - c_n| > (1-\varepsilon)t$ on $\cup B_{n,i}(t)$, and the inequality of (2.10) follows.

The equality of (2.11) follows from the disjointness of the sets involved, while the inequality follows from (2.14). The equality (2.12) is a consequence of the independence of the sets $A_{n,i}(t)$ and $\{|X_{n,i}| > t\}$. The inequality (2.13) follows from the inclusion $A_{n,i}(t) \supset C_n(\varepsilon t/2)$.

From the convergence $S_n - c_n \xrightarrow{\mathscr{D}} S$ and from Lemma 1.3 (i) it follows that for a fixed $0 < \varepsilon < 1$, $P\{C_n(\varepsilon t/2)\} \to 1$ as $t \to \infty$ uniformly in n; hence by (2.10)–(2.13), for any $0 < \varepsilon < 1$ there exists a number t_ε such that $Q_n\{(1-\varepsilon)t\} \geqq (1-\varepsilon)^{\alpha+1} R_n(t)$ for all $n \geqq 1$, $t > t_\varepsilon$, $0 < \varepsilon < 1$, and $Q \geqq \underset{\sim}{R}$ and $\bar{Q} \geqq \bar{R}$ follow readily. The proof of Theorem 0 is complete.

§3. Applications

In the following theorems, $S_n = \sum_{i=1}^n X_i$.

THEOREM 3.1. *Let $\{X_i, i \geqq 1\}$ be a sequence of independent random variables which are median-bounded, and let $S_n - c_n \to S$ a.s. Let f be a weak moment function, and let*

$$\rho = \lim_{t\to\infty} \sum_{i=1}^{\infty} \int_{|X_i|>t} f(X_i).$$

If $\rho = 0$, then $Ef(S_n - c_n) \to Ef(S) < \infty$, and $Ef(S - S_n + c_n) \to 0$. If $\rho = \infty$, then $Ef(S_n - c_n) \to Ef(S) = \infty$.

REMARK. It is clear that ρ can only equal 0 or ∞.

PROOF. We consider two arrays. It may be verified that Theorem 0 applies to each.

The first array is $\{X_{n,i} = X_i, 1 \leqq i \leqq n, n \geqq 1\}$. Its nth row sum is the S_n defined above, so $S_n - c_n \xrightarrow{\mathcal{D}} S$. Computing the expression in (0.2), we see that R exists and equals ρ.

The second array is $\{X^*_{n,i}; 1 \leqq i \leqq 2, n \geqq 1\}$, where $X^*_{n,1} = S_n - c_n$, $X^*_{n,2} = S - S_n + c_n$. Taking $c^*_n = 0$, we have $S^*_n - c^*_n = S$, $n \geqq 1$. The expression corresponding to (0.2) is

$$R^* = \lim_{t\to\infty} \overline{\lim_{n\to\infty}} \left[\int_{|S_n - c_n|>t} f(S_n - c_n) + \int_{|S - S_n + c_n|>t} f(S - S_n + c_n) \right]. \tag{3.1}$$

The following reasoning is supported by the conclusions of Theorem 0.

Suppose that $\rho = R = 0$. Then $Ef(S_n - c_n) \to Ef(S) < \infty$. We then trivially have $Ef(S^*_n - c^*_n) \to Ef(S) < \infty$ as well, and it follows that R^* exists and equals 0. In particular, $\lim_{t\to\infty}\overline{\lim}_{n\to\infty}$ of the second integral in (3.1) equals 0. Since $S - S_n + c_n \to 0$ a.s., it easily follows that $Ef(S - S_n + c_n) \to 0$.

Next suppose that $\rho = R = \infty$. Then $Ef(S_n - c_n) \to \infty$, and therefore $\lim_{t\to\infty}\underline{\lim}_{n\to\infty}$ of the first integral in (3.1) equals ∞. It immediately follows that R^* exists and equals ∞, and hence $Ef(S^*_n - c^*_n) \to \infty$, i.e., $Ef(S) = \infty$.

THEOREM 3.2. *Let $\{X_i, i \geqq 1\}$ be a sequence of independent, identically distributed random variables.*

(i) *Let $0 < \alpha < 2$ and $E|X_i|^\alpha < \infty$. Then $E|n^{-1/\alpha}S_n|^\alpha \to 0$.*

(ii) *Let $a_n > 0$ and b_n be constants such that $a_n^{-1}(S_n - c_n) \xrightarrow{\mathcal{D}} S$, where S has a stable distribution of exponent $0 < \alpha \leqq 2$. Then for any $0 < \beta < \alpha$, $E|a_n^{-1}(S_n - b_n)|^\beta \to E|S|^\beta < \infty$.*

REMARK. A special case of (ii) was proved by Owen [3], who required normal attraction, i.e., $a_n = n^{1/\alpha}$,

PROOF. (i) By Kolmogorov and Marcinkiewicz [2], $n^{-1/\alpha}S_n \to 0$ a.s. We may apply Theorem 0 to the array $\{X_{n,i} = n^{-1/\alpha}X_i;\ 1 \leqq i \leqq n, n \geqq 1\}$, with $c_n = 0$ and $f(x) = |x|^\alpha$. The sum in (0.2) equals $\int |X_1|^\alpha I(|X_1| > tn^{1/\alpha})$. It immediately follows that R exists and equals 0, and the convergence follows.

(ii) We apply Theorem 0 to the array $\{X_{n,i} = a_n^{-1}X_i;\ 1 \leqq i \leqq n, n \geqq 1\}$ with $c_n = a_n^{-1}b_n$ and $f(x) = |x|^\beta$. Denoting the sum in (0.2) by $R_n(t)$, we have

$$R_n(t) = na_n^{-\beta}\int_{|x|>ta_n} |x|^\beta dF(x)$$

where F is the distribution function of X_1. Let

$$\mu(t) = \int_{|x|\leqq t} x^2 dF(x).$$

From page 579 of [1], we have that for some $c > 0$

$$na_n^{-2}\mu(ta_n) \to ct^{2-\alpha}, \qquad n \to \infty$$

and that

$$[s^{2-\beta}/\mu(s)]\int_{|x|>s} |x|^\beta dF(x) \to r, \qquad s \to \infty,$$

where $r = (\alpha - \beta)^{-1}(2 - \alpha)$. Replacing s by ta_n and multiplying together the two convergences yields

$$R_n(t) \to crt^{\beta-\alpha}, \qquad n \to \infty.$$

Letting $t \to \infty$ yields $R = 0$, and the convergence in (ii) follows.

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